Complex Wavenumber Rossby Wave Ray Tracing

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ABSTRACT

We present a methodology for performing complex wavenumber Rossby wave ray tracing in a two-dimensional zonally asymmetric flow. Both wave trajectory and amplitude are calculated. This ray tracing framework is first derived using perturbation methods. The amplitude calculation is then validated in a simple idealized system for which exact solutions can be calculated. We then apply this ray tracing to both solid body rotation on a sphere and observed upper tropospheric climatological fields. These ray tracing solutions are compared with similarly forced solutions of the linearized barotropic vorticity equation (LBVE). Both real and complex wavenumber ray tracings follow trajectories matched by LBVE solutions. Complex wavenumber ray tracings on observed atmospheric fields at times follow trajectories distinct from real wavenumber Rossby waves. For example, complex wavenumber ray tracings initiated over the eastern equatorial Pacific Ocean during boreal summer propagate northeastward into the subtropics over the Atlantic Ocean and southeastward. Similarly initiated real wavenumber ray tracings stay within the deep tropics and propagate westward. We find these complex wavenumber Rossby wave trajectories and their amplitudes are generally consistent with LBVE solutions and indicate that this methodology can identify Rossby wave effects distinct from traditional real wavenumber tracings.

1. Introduction

Ray tracing is often used to explore the energy propagation of Rossby waves with stationary or near-stationary phase speeds. The ray trajectories indicate how information is communicated through the atmosphere over large distances, as well as the time-scales over
which this information is conveyed. Rossby wave ray tracing has provided insight into the atmospheric response to steady thermal and orographic forcing (Hoskins and Karoly 1981), the response to low-frequency forcing (Li and Nathan 1994), tropical-extratropical interactions (Hoskins and Karoly 1981), interactions within the extratropics (Wang et al. 2007), and propagation within the tropics of easterly waves (Sobel and Bretherton 1999).

Heretofore, investigations have been restricted to Rossby waves with real wavenumbers. However, thermal or orographic forcing of the atmosphere will in theory produce large-scale waves with both complex and real wavenumbers. Rossby waves with complex wavenumbers will possess amplitudes that are modified, as they propagate, by the imaginary component of their wavenumber. Previous work has suggested that such modulated waves, given either a slowly decaying amplitude or growth through interaction with the mean flow, may persist and be of import for the communication of information over large distances within the atmosphere (Dickinson and Clare 1973).

Here we present a method for tracing both the trajectory and amplitude of complex wavenumber barotropic Rossby waves in a zonally asymmetric two-dimensional environment. In Section 2 we provide theoretical context for the complex ray tracing of wave trajectories. We expressly derive equations for ray tracing two-dimensional stationary barotropic Rossby waves with complex wavenumbers in Section 3. In Section 4 we apply this complex Rossby wave ray tracing to an idealized environment and show that this ray tracing amplitude matches exact analytic solutions. We then perform complex ray tracing on a sphere in solid body rotation and compare these tracings to forced solutions of the linearized barotropic vorticity equation (LBVE) (Section 5). Complex ray tracing is then performed on realistic two-dimensional atmospheric flows and compared with forced solutions of the LBVE (Section
2. Ray Tracing with Complex Wavenumbers

a. Ray Equations

The tracing of wave trajectories follows the general theory of ray tracing in an anisotropic medium (Whitham 1974; Lighthill 1978). Specifically, we consider a two-dimensional, approximate plane wave solution with slowly varying amplitude, wavenumber and position of the form \( A \exp[i\phi(x, y, t)] \), where \( A \) is the wave amplitude, and the wave phase \( \phi \) can be expressed locally as

\[
\phi(x, y, t) \approx kx + ly - \omega t,
\]  

(2.1)

where \( \omega(x, y, t) = -\partial \phi / \partial t \) is the slowly varying frequency, and \( k(x, y, t) = \partial \phi / \partial x \) and \( l(x, y, t) = \partial \phi / \partial y \) are slowly varying wavenumbers in the zonal and meridional directions, respectively. Equality of the mixed partial derivatives then implies that the changes in \( k \) and \( l \) must satisfy

\[
\frac{dk}{dt} = -\frac{\partial \omega}{\partial x},
\]  

(2.2a)

\[
\frac{dl}{dt} = -\frac{\partial \omega}{\partial y},
\]  

(2.2b)

along wave rays defined by

\[
\frac{dx}{dt} = u_g = \frac{\partial \omega}{\partial k},
\]  

(2.3a)

\[
\frac{dy}{dt} = v_g = \frac{\partial \omega}{\partial l},
\]  

(2.3b)
where the group velocity vector $\mathbf{c}_g = (u_g, v_g)$. With a dispersion relation $\omega_0 = \omega(k, l; x, y)$ determined from the local dynamics, equations 2.2 and 2.3 provide a complete set of ODEs for $(x, y, k, l)$ from which the propagation of wave energy along a ray may be traced.

Typically, to perform stationary wave tracing, rays are initiated at some location $(x, y)$ for a prescribed integer zonal wavenumber $k$, and the initial meridional wavenumber $l$ is then solved for using the dispersion relation (i.e. $\omega(k, l, x, y) = \omega_0 = 0$). Depending on the dispersion relation this procedure may yield multiple initial values for $l$, some of which may be complex. Ray tracing is usually only performed for real initial meridional wavenumber solutions; however, waves with complex dispersion relations, complex wavenumbers and complex group velocities have been analyzed in fields such as geometric optics and plasma physics, and have been used in geophysical fluid dynamics to describe convective and absolute instabilities of baroclinic eddies (Merkine 1977; Merkine and Shafranek 1980).

Muschietti and Dum (1993) and Sonnenschein et al. (1998) examined the propagation of waves in dissipative medium, where the solution is also decaying, as it may be in dispersive medium such as the inviscid rotating atmosphere considered here. Their approach was to let wavenumbers vary in complex space, but restrict variations of the ray coordinates to real coordinate space by considering only the real part of the group velocity. This approach, based on saddle point analysis, is similar to the handling suggested by Sommerfeld (1914) and Brillouin (1914) for light traveling in absorptive medium, as well as the analysis of Merkine and Shafranek (1980) for the temporal and spatial evolution of unstable baroclinic disturbances.

For this work, we take a slightly different approach and derive a framework for tracing complex wavenumber waves using perturbation methods. For a one-dimensional system with
real wavenumber \( k \), Equation (2.2a) simplifies to

\[
\frac{dk}{dt} = \frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial x} = -\frac{\partial \omega}{\partial x}.
\]

(2.4)

Suppose now that \( k \) is complex, such that \( k = k_r + ik_i \). The dispersion relation and thus the group velocity will consequently be complex. The imaginary component of the wavenumber is readily interpretable as a modulation of plane wave amplitude (i.e. \( A \exp(-k_i x) \exp(ik_r x - \omega t) \)); however, the complex group velocity leads to an imaginary trajectory, or characteristic, for which the physical interpretation is unclear.

When \( k \) is complex, but \( x \) is required to remain real, the one-dimensional analog of Equation 2.4 devolves into a pair of equations for the real and imaginary parts of \( k \):

\[
\frac{\partial k_r}{\partial t} + u_{gr} \frac{\partial k_r}{\partial x} = -\frac{\partial \omega}{\partial x} \bigg|_r + u_{gi} \frac{\partial k_i}{\partial x}
\]

(2.5a)

\[
\frac{\partial k_i}{\partial t} + u_{gr} \frac{\partial k_i}{\partial x} = -\frac{\partial \omega}{\partial x} \bigg|_i - u_{gi} \frac{\partial k_r}{\partial x}
\]

(2.5b)

where the subscripts \( r \) and \( i \) indicate real and imaginary parts, respectively. It can be seen by, for example, cross-differentiating the first and last terms of (2.5ab) to eliminate \( k_i \) in favor of \( k_r \) that this pair of equations has an essentially elliptic character, and thus cannot be solved exactly along real characteristics.

Our goal is to find an approximate solution to (2.5) along real characteristics, despite the elliptic character of the full equations. Both the real and imaginary wavenumber equations (2.5) are expressed as an equality of the local change in time of the wavenumber plus the advection of the wavenumber gradient by the real group velocity (lhs terms) with the local change of either the real (2.5a) or imaginary (2.5b) phase with respect to local wavenumber plus the advection of the wavenumber gradient by the imaginary group velocity (rhs terms).
The first 3 terms of these equations are identical in form to Equation 2.4, and represent the change in the wavenumber along the real characteristic. However, the last terms, which give the advection of the wavenumber gradient by the imaginary group velocity, are new.

As we do not wish to follow ray paths that stray into complex space we must eliminate or otherwise determine these last two terms. To do so we assume, per perturbation theory, that $k_i$ is $\mathcal{O}(\alpha)$ relative to $k_r$, where $\alpha \ll 1$ and we may take $k_r$ to be $\mathcal{O}(1)$. Physically, this assumption implies a slow variation of the wave amplitude on a scale much longer than the wavelength. The first 3 terms of Equation 2.5a are then $\mathcal{O}(1)$, but the last term is $\mathcal{O}(\alpha^2)$ and can be dropped, which reduces this equation to the form of Equation 2.4. However, all 4 terms of Equation 2.5b are $\mathcal{O}(\alpha)$, so none can be eliminated. We therefore need to find an explicit way of describing the last term of this equation. To do so, we need an expression for $\partial k_r / \partial x$. Such an expression can be derived by taking the gradient of the $\mathcal{O}(1)$ terms in Equation 2.5a.

\[ \text{b. Wavenumber Gradient Equations} \]

We now develop an additional expression that represents the evolution of $\partial k_r / \partial x$ along the ray. Taking the partial derivative of (2.5a) with respect to $x$ after neglecting the $\mathcal{O}(\alpha^2)$ term and while holding $t$ fixed, we obtain

\[ \frac{\partial}{\partial t} \left( \frac{\partial k_r}{\partial x} \right) + u_g \frac{\partial}{\partial x} \left( \frac{\partial k_r}{\partial x} \right) + \frac{\partial u_g}{\partial x} \frac{\partial k_r}{\partial x} = - \frac{\partial}{\partial x} \left( \frac{\partial \omega}{\partial x} \right) \bigg|_r \]  

(2.6)

and then make the additional approximations

\[ \frac{\partial u_g}{\partial x} = \frac{\partial}{\partial x} \frac{\partial \omega}{\partial k} \bigg|_r \approx \frac{\partial^2 \omega}{\partial x \partial k} \bigg|_r + \frac{\partial^2 \omega}{\partial k^2} \bigg|_r \frac{\partial k_r}{\partial x} \]  

(2.7)
and

\[
\frac{\partial}{\partial x} \frac{\partial \omega}{\partial x} \approx \frac{\partial^2 \omega}{\partial x \partial k} \bigg|_r \frac{\partial k_r}{\partial x} + \frac{\partial^2 \omega}{\partial x^2} \bigg|_r, \tag{2.8}
\]

in which \(O(\alpha, \alpha^2)\) terms are neglected in the chain-rule expressions involving the complex \(k(x)\). The first two terms of (2.6) are the derivative \(d(\partial k_r/\partial x)/dt\) along the real ray \(dx/dt = u_g\), while the rest can be written in terms of quantities evaluated on the ray.

Thus, our approximate complex wavenumber ray tracing equations are the extended set

\[
\frac{dk_r}{dt} = -\frac{\partial \omega}{\partial x} \bigg|_r \tag{2.9a}
\]

\[
\frac{dk_i}{dt} = -\frac{\partial \omega}{\partial x} \bigg|_i + u_g \frac{\partial k_r}{\partial x} \tag{2.9b}
\]

\[
\frac{d}{dt} \left( \frac{\partial k_r}{\partial x} \right) = -\frac{\partial^2 \omega}{\partial k^2} \bigg|_r \left| \frac{\partial k_r}{\partial x} \right|^2 \bigg|_r - 2 \frac{\partial^2 \omega}{\partial k \partial x} \frac{\partial k_r}{\partial x} - \frac{\partial^2 \omega}{\partial x^2} \bigg|_r \tag{2.9c}
\]

\[
\frac{dx}{dt} = u_{g_r} = \frac{\partial \omega}{\partial k} \bigg|_r \tag{2.9d}
\]

In 2.9d, only the real part of the group velocity appears, giving a real ray trajectory, as the instantaneous value of the imaginary group velocity \(u_{g_i}\) along the ray path is utilized in (2.9b) but does not directly contribute to determining the ray path. The critical approximation that allows the formulation (2.9) and solution along the real ray is the neglect of the last term in (2.5a) to obtain (2.9a).

c. Two-Dimensional Form

When \(k\) and \(l\) are complex, but \(x\) and \(y\) are required to remain real, the two-dimensional ray tracing equations (2.2) devolve into a similar form as (2.5):

\[
\frac{dk_r}{dt} = \frac{\partial k_r}{\partial t} + u_{g_r} \frac{\partial k_r}{\partial x} + v_{g_r} \frac{\partial k_r}{\partial y} = -\frac{\partial \omega}{\partial x} \bigg|_r + u_{g_i} \frac{\partial k_i}{\partial x} + v_{g_i} \frac{\partial k_i}{\partial y} \tag{2.10a}
\]
\[ \frac{dl_r}{dt} = \frac{\partial l_r}{\partial t} + u_g \frac{\partial l_r}{\partial x} + v_g \frac{\partial l_r}{\partial y} = -\frac{\partial \omega}{\partial y} + u_g \frac{\partial l_i}{\partial x} + v_g \frac{\partial l_i}{\partial y} \]  
(2.10b)

\[ \frac{dk_i}{dt} = \frac{\partial k_i}{\partial t} + u_g \frac{\partial k_i}{\partial x} + v_g \frac{\partial k_i}{\partial y} = -\frac{\partial \omega}{\partial x} - u_g \frac{\partial k_r}{\partial x} - v_g \frac{\partial k_r}{\partial y} \]  
(2.10c)

\[ \frac{dl_i}{dt} = \frac{\partial l_i}{\partial t} + u_g \frac{\partial l_i}{\partial x} + v_g \frac{\partial l_i}{\partial y} = -\frac{\partial \omega}{\partial y} - u_g \frac{\partial l_r}{\partial x} - v_g \frac{\partial l_r}{\partial y} \]  
(2.10d)

As for the 1-D system, when \( k_i \) and \( l_i \) are \( O(\alpha) \) with \( \alpha \ll 1 \) we can neglect the last 2 \( O(\alpha^2) \) term of Equations 2.10a and b, but we cannot drop the last 2 terms of Equations 2.10c and d. As before, we derive explicit expressions for \( \frac{\partial k_r}{\partial x} \), \( \frac{\partial k_r}{\partial y} \), \( \frac{\partial l_r}{\partial x} \), \( \frac{\partial l_r}{\partial y} \), using the \( O(1) \) terms of Equations 2.10a and b:

\[ \frac{dk_r}{dt} = \frac{\partial k_r}{\partial t} + \frac{\partial \omega}{\partial k} \frac{\partial k_r}{\partial x} + \frac{\partial \omega}{\partial l} \frac{\partial l_r}{\partial y} = -\frac{\partial \omega}{\partial x} \]  
(2.11a)

\[ \frac{dl_r}{dt} = \frac{\partial l_r}{\partial t} + \frac{\partial \omega}{\partial k} \frac{\partial l_r}{\partial x} + \frac{\partial \omega}{\partial l} \frac{\partial l_r}{\partial y} = -\frac{\partial \omega}{\partial y} \]  
(2.11b)

Specifically, \( \frac{\partial}{\partial x} \) of Equation 2.11a gives

\[ \frac{d}{dt} \left( \frac{\partial k_r}{\partial x} \right) = \frac{\partial}{\partial t} \left( \frac{\partial k_r}{\partial x} \right) + u_g \frac{\partial}{\partial x} \left( \frac{\partial k_r}{\partial k} \right) + v_g \frac{\partial}{\partial y} \left( \frac{\partial k_r}{\partial x} \right) \]

\[ = -2 \frac{\partial^2 \omega}{\partial k \partial x} \frac{\partial k_r}{\partial x} - \frac{\partial^2 \omega}{\partial k^2} \left| \frac{\partial k_r}{\partial x} \right|^2 - 2 \frac{\partial^2 \omega}{\partial l \partial k} \frac{\partial l_r \partial k_r}{\partial x \partial x} \]

\[ - 2 \frac{\partial^2 \omega}{\partial l \partial x} \frac{\partial l_r}{\partial x} - \frac{\partial^2 \omega}{\partial l^2} \left| \frac{\partial l_r}{\partial x} \right|^2 - \frac{\partial^2 \omega}{\partial x^2} \]  
(2.12)

Equation 2.12 is the approximate equation for the zonal gradient of \( k \) along the ray (2.3).

Three more equations are needed to completely define the wavenumber gradients needed for the additional terms in the 2-D imaginary wavenumber equations (Equations 2.10c and d). All the terms on the right-hand side of Equation 2.12 can be evaluated along the ray. In (2.12), the relation \( \partial k / \partial y = \partial^2 \phi / \partial y \partial x = \partial l / \partial x \) has been used.

The three remaining wavenumber gradient equations are derived similarly, by taking \( \partial / \partial y \) of Equation 2.11a, and \( \partial / \partial x \) and \( \partial / \partial y \) of Equation 2.11b:
of the waves will be controlled by the form of the dispersion relation \( \omega = \omega(k, l, x, y) \), where \( \omega_0 \) is the wave frequency.

Equations 2.12-2.15 are the 4 wavenumber gradient equations needed to solve for the last 2 terms of the Equations 2.10c and d. The approximate ray-tracing equations for the two-dimensional complex waves are then (2.11), (2.10cd), (2.12)-(2.15), and (2.3).
d. Amplitude Evolution for Stationary Waves

For stationary waves, which have $\omega_0 = 0$, the local variation in the amplitude of the wavepacket near a point $x_0 = (x_0, y_0)$ is described approximately by

$$|A(x)| = |A(x_0)| \exp\left[k_i(\epsilon x) \cdot (x - x_0)\right], \quad x \approx x_0$$

where $k = (k, l)$, $x = (x, y)$, and $\epsilon \ll 1$. The local change in wave amplitude is then

$$\frac{d|A|}{dx}(x) = \left[k_i(\epsilon x) + \epsilon \frac{dk_i}{dx}(\epsilon x) \cdot (x - x_0)\right]|A(x)|, \quad x \approx x_0$$

(2.16)

As $x_0$ is arbitrary in space and the term proportional to $\epsilon$ in Equation 2.16 is arbitrarily small near $x_0$, we may integrate Equation 2.16 along the ray to obtain

$$|A(x)| \approx |A(0)| \exp\left[\int_0^x k_i(\epsilon x') \cdot dx'\right]$$

(2.17)

This expression may give either wave decay or wave growth, where the latter may indicate instability of the background flow.

3. Barotropic Rossby Waves

Equations 2.10c and d, 2.11, and 2.12-2.15, along with

$$\frac{dx}{dt} = u_g, \quad \frac{\partial \omega}{\partial k}
$$

(3.1a)

$$\frac{dy}{dt} = v_g, \quad \frac{\partial \omega}{\partial l}
$$

(3.1b)
provide the complete set of two-dimensional complex wavenumber ray tracing equations for this approximation. For two-dimensional barotropic Rossby waves the dispersion relation is:

\[
\omega(k, l, x, y) = \bar{u}_M k + \bar{v}_M l + \frac{l \partial \bar{q} / \partial x - k \partial \bar{q} / \partial y}{k^2 + l^2}
\]  

(3.2)

where

\[
(\bar{u}_M, \bar{v}_M) = \left(\frac{\bar{u}, \bar{v}}{\cos \theta}\right)
\]

is the Mercator projection of the time-mean zonal and meridional winds, \(\theta\) is latitude, and

\[
\bar{q} = f_0 + \beta y + \nabla^2 \bar{\psi}
\]

is the time-mean absolute vorticity.

Using this dispersion relation and Equation 2.11 gives the following ray-tracing equations for the real parts of the wavenumbers:

\[
\frac{dk_r}{dt} = \text{Re} \left( -k \frac{\partial \bar{u}_M}{\partial x} - l \frac{\partial \bar{v}_M}{\partial x} + \frac{k \partial^2 \bar{q} / \partial x \partial y - l \partial^2 \bar{q} / \partial x^2}{k^2 + l^2} \right)
\]

(3.3a)

\[
\frac{dl_r}{dt} = \text{Re} \left( -k \frac{\partial \bar{u}_M}{\partial y} - l \frac{\partial \bar{v}_M}{\partial y} + \frac{k \partial^2 \bar{q} / \partial y^2 - l \partial^2 \bar{q} / \partial x \partial y}{k^2 + l^2} \right)
\]

(3.3b)

along the real rays (from Equation 3.1):

\[
\frac{dx}{dt} = u_{gr} = \text{Re} \left( \frac{\partial \omega}{\partial k} \right) = \text{Re} \left( \bar{u}_M + \frac{(k^2 - l^2) \partial \bar{q} / \partial y - 2 kl \partial \bar{q} / \partial x}{(k^2 + l^2)^2} \right)
\]

(3.4a)

\[
\frac{dy}{dt} = v_{gr} = \text{Re} \left( \frac{\partial \omega}{\partial l} \right) = \text{Re} \left( \bar{v}_M + \frac{2 kl \partial \bar{q} / \partial y + (k^2 - l^2) \partial \bar{q} / \partial x}{(k^2 + l^2)^2} \right).
\]

(3.4b)
Using the above expressions, we can form the ray-tracing equations for $\partial k_r/\partial x$, $\partial k_r/\partial y$, $\partial l_r/\partial x$, and $\partial l_r/\partial y$, and thus also Equations 2.10c and d for $k_i$ and $l_i$. 
4. Analytical Example: Amplitude Variation

We next apply this two-dimensional complex wavenumber Rossby wave ray tracing to a simple, idealized setting for which an exact analytic solution of wave amplitude can also be calculated. We employ an infinite $\beta$-plane that for the region $0 < y < Y$ additionally has bottom topography, $\beta_T$. Thus, there are 3 regions: $y < 0$, $0 < y < Y$ and $y > Y$ (I, II, and III, respectively). Conditions are imposed such that the first and last regions (I and III) only support real Rossby waves; the middle section (region II), given a specific range of bottom topographies, creates a meridional gradient of potential vorticity ($\beta + \beta_T$) that supports only evanescent wave modes, which are represented as complex wavenumber Rossby waves (Figure 1).

Throughout the entire domain (Regions I, II and III), the zonal flow $\bar{u}_M$ is constant, the meridional flow $\bar{v}_M$ is zero and the zonal gradient of absolute vorticity ($\partial \bar{q}/\partial x$) is constant. For continuity to be met at all time and all values of $x$ the wave frequency and zonal wavenumber must match at the two boundaries ($y = 0$ and $y = Y$), so $\omega$ and $k$ are constant everywhere. The same cannot be claimed for the meridional wavenumbers, due to the variation of the medium in the $y$-direction. Thus, the solution for the wave streamfunction $\psi_X$ in region $X = \{I, II, III\}$ has the form

$$\psi_X = \sum_{j=1}^{J_X} A_{Xj} \exp[i(kx + l_{Xj}y - \omega t)], \quad (4.1)$$

where $J_X$ is the number of wave modes in region $X$, and $\omega$ and $k$ are independent of the region.

For this system, Regions I and III support two real wave modes for the prescribed $k$, $u$, $\beta$, $\partial \bar{q}/\partial x$, $v$ and $\omega_0$. The two waves possess real meridional wavenumber of opposite sign and
thus propagate in opposite directions (one north, one south). For the problem we initiate a
single, northward propagating Rossby wave with amplitude 1 within Region I. Upon reaching
the $y = 0$ bound two transmitted waves and a reflected wave are generated. The reflected
wave is the southward propagating Rossby wave supported by Region I.

For Region II, two Rossby wave modes with complex conjugate meridional wavenumbers
are supported. The imaginary part of the meridional wavenumber modulates the wave am-
plitude; one mode is decaying, the other is growing. By design, the solutions have positive
real meridional wavenumber components, such that both waves propagate northward. Con-
sequently, no reflection is supported when these waves reach the $y = Y$ boundary (given the
prescribed $\bar{u}_M, \bar{v}_M, \beta, \beta_T, \partial \bar{q}/\partial x$, and $\omega_0$.

For Region III, only the northward wave is spawned by incidence of the complex waves
at $y = Y$. Thus at $y = 0$, we have 4 waves to consider, the Region I incident northward
propagating real wave, the Region I reflected southward propagating real wave, and the
2 Region II northward propagating complex waves (Figure 1). These 4 waves satisfy two
boundary conditions at $y = 0$, discontinuity matching of the quasi-geostrophic potential
vorticity equation and continuity of pressure across the boundary (Appendix).

At $y = Y$, we have only 3 waves to consider, the 2 northward propagating Region II
complex waves and the Region III northward propagating real wave (Figure 1). These 3
waves must satisfy the same 2 boundary conditions at $y = Y$. Thus, the system requires
that

$$A_{In} + A_{Is} = A_{IIg} + A_{IId} \quad \text{at } y = 0 \quad (4.2a)$$

$$\frac{\partial A_{In}}{\partial y} + \gamma A_{In} + \frac{\partial A_{Is}}{\partial y} + \gamma A_{Is} = \frac{\partial A_{IIg}}{\partial y} + \gamma A_{IIg} + \frac{\partial A_{IId}}{\partial y} + \gamma A_{IId} \quad \text{at } y = 0 \quad (4.2b)$$
\[ A_{III} = A_{IIg} + A_{IId} \quad \text{at } y = Y \quad (4.2c) \]

\[ \frac{\partial A_{III}}{\partial y} + \gamma A_{III} = \frac{\partial A_{IIg}}{\partial y} + \gamma A_{IIg} + \frac{\partial A_{IId}}{\partial y} + \gamma A_{IId} \quad \text{at } y = Y \quad (4.2d) \]

where the subscripts \( In, Is, IIg, IId \), and \( III \) denote the Region I northward, Region I southward, Region II growing, Region II decaying and Region III northward propagating Rossby waves, respectively, and \( \gamma = \bar{q}_x/uk \).

\( A_{In} \) is prescribed with amplitude 1. We solve for the 4 other waves given the above system \((4.2)\). The solutions (see Appendix) give the amplitude of each wave, which varies for a given background and Region I zonal wavenumber Rossby wave as a function of Region II domain size.

We then compare this amplitude solution to a complex wavenumber ray tracing of Region II from \( y = 0 \) to \( y = Y \). Within Region II the dispersion relation, \((3.2)\), reduces to

\[ \omega = \bar{u}_M k + \frac{l \partial \bar{q}/\partial x}{k^2 + l^2} \quad (4.3) \]

the ray tracing equations are

\[ \frac{dk}{dt} = - \frac{\partial \omega}{\partial x} = 0 \quad (4.4a) \]

\[ \frac{dl}{dt} = - \frac{\partial \omega}{\partial y} = 0 \quad (4.4b) \]

and

\[ \frac{dx}{dt} = u_g = Re \left( \frac{\partial \omega}{\partial k} \right) = Re \left( \bar{u}_M + \frac{(k^2 - l^2) \partial \bar{q}/\partial y - 2kl \partial \bar{q}/\partial x}{(k^2 + l^2)^2} \right) \quad (4.5a) \]

\[ \frac{dy}{dt} = v_g = Re \left( \frac{\partial \omega}{\partial l} \right) = Re \left( \frac{2kl \partial \bar{q}/\partial y + (k^2 - l^2) \partial \bar{q}/\partial x}{(k^2 + l^2)^2} \right) \quad (4.5b) \]

respectively, and the evolution of the wave amplitude along the ray is described by

\[ \frac{dA}{dy} = -l_i A \quad (4.6) \]
As Equation 4.4 makes clear within Region II there is no change of the Rossby wave wavenumbers, either real or complex; rather, this simple analytic system enables comparison of calculated and ray traced complex wavenumber Rossby wave amplitudes.

Figure 2 shows how wave height solutions vary as a function of $Y$ for $\bar{u}_M = 20m/s$, $\bar{v}_M = 0m/s$, $\beta = 2 \times 10^{-11} s^{-1} m^{-1}$, $\beta + \beta_T = -3.26 \times 10^{-11} s^{-1} m^{-1}$, $\partial q/\partial x = -4.5 \times 10^{-11} s^{-1} m^{-1}$, $\omega_0 = 0$, and zonal wavenumber $k = 7.85 \times 10^{-7} m^{-1}$. The wave height is the real part of the wave function, the amplitude times the wave phase, e.g. $Re\{\psi_{II}\} = Re\{\Sigma A_{IIj} \exp (l_{IIj}y) \exp (l_{IIj}y - \omega_0 t)\}$ in region II. Wave heights are shown at $y = Y$ as this quantity accounts for wave phase and is the quantity matched in the system solution at this location (see Appendix). At $y = Y$, the Region III Rossby wave height closely matches the height of the Region II growing mode.

Figure 3 shows the amplitude of the Region II decaying mode at $y = Y$, as well as the amplitude of this mode calculated numerically via the complex wavenumber ray-tracing amplitude equation (4.6). The two solutions match well, showing that complex wavenumber ray tracing appropriately captures the modulation of wave amplitude, as well as wave propagation characteristics, which define this modulation. Growing solutions also match (not shown). This analysis was repeated for a number of backgrounds and initial zonal wavenumbers; in all cases the ray tracing and analytic solutions matched.

5. Solid Body Rotation

We next performed stationary barotropic Rossby wave ray tracing of both complex and real wavenumber waves on a variety of two-dimensional fields. Ray tracing trajectories
were validated through comparison with forced solutions of the barotropic vorticity equation (LBVE) linearized about the same two-dimensional fields, as per Sham and Tziperman (2007).

We first experimented with solid body rotation where \( \bar{u}_M \) is constant, \( \bar{v}_M = 0 \) and \( \bar{q}_x = 0 \). Thus, the two-dimensional dispersion relation reduces to

\[
\omega = \bar{u}_M k + \frac{-\bar{q}_y k}{k^2 + l^2}
\]

(5.1)

The ray tracing equations are again 2.10c and d, 2.11-2.15 and 3.1. In this instance, the real wavenumber and group velocity equations are somewhat simplified to

\[
\frac{dk_r}{dt} = -\frac{\partial \omega}{\partial x} = 0 \quad (5.2a)
\]

\[
\frac{dl_r}{dt} = -\frac{\partial \omega}{\partial y} = \frac{k\partial^2 \bar{q}/\partial y^2}{(k^2 + l^2)} \quad (5.2b)
\]

and

\[
\frac{dx}{dt} = u_{gr} = Re \left( \frac{\partial \omega}{\partial k} \right) = Re \left( \bar{u}_M + \frac{(k^2 - l^2)\partial \bar{q}/\partial y}{(k^2 + l^2)^2} \right) \quad (5.3a)
\]

\[
\frac{dy}{dt} = v_{gr} = Re \left( \frac{\partial \omega}{\partial l} \right) = Re \left( \frac{2kl\partial \bar{q}/\partial y}{(k^2 + l^2)^2} \right) \quad (5.3b)
\]

respectively. The evolution of the wave amplitude is described by Equation 2.17. The imaginary wavenumber expression includes the additional terms seen in Equations 2.10c and d.

We ran these solid body rotation experiments using \( \bar{u}_M = 15 m/s \). At first, to insure that \( \alpha \ll 1 \) (i.e. \( k_i \ll k_r \) and \( l_i \ll l_r \)), we initialized the complex wavenumber ray tracings with an apportioned complex zonal wavenumber, e.g. \( k = 5 + 0.01i \), such that, given background flow, the meridional wavenumber was similarly scaled among its real and imaginary components.

Figure 4 presents the ray tracing integrations for four real wavenumber stationary Rossby
waves initiated with zonal wavenumber $k = 5$ and four complex wavenumber stationary Rossby waves initiated with zonal wavenumber $k = 5 + 0.01i$. These real and complex wavenumber ray tracings are pairwise initiated at the same locations and are superimposed on the steady solution of the LBVE model in response to forcing centered at $0^\circ$N, 180°E. The real and complex wavenumber trajectories correspond with one another, though they do not match precisely. The small imaginary complex wavenumber introduces slight changes in amplitude and trajectory over the course of the ray integration.

These differences can be seen more clearly in plots of the evolution of the wavenumbers and amplitude of one real wavenumber stationary Rossby wave ray tracing and the corresponding complex wavenumber stationary Rossby wave ray tracing, initiated at the same location (Figure 5). For this background, the ray tracing initiated with zonal wavenumber $k = 5$ has a real initial meridional wavenumber. As both wavenumbers are real there is no change in amplitude during the ray tracing integration. The ray tracing initiated with zonal wavenumber $k = 5 + 0.01i$, however, produces a slight decay of amplitude during integration; this small amplitude decay reflects the small magnitude of the imaginary wavenumbers. During the 15-day integration these imaginary wavenumbers remain several orders of magnitude smaller than the real wavenumbers, consistent with our assumption that $\alpha \ll 1$. Both the real and complex wavenumber trajectories match the steady solution of the LBVE model (Figure 4).

A number of the complex wavenumber ray tracings initiated with $k = 5 + 0.01$ behaved erratically and diverged from their real wavenumber counterparts (Figure 6 and 7). This divergence occurred when the imaginary wavenumber component grew to magnitudes equal to that of the real wavenumber component, i.e. $\alpha \approx 1$. At that point, the wave trajectory
would devolve to a primarily zonal propagation and the wave amplitude would shift radically. This erratic behavior also occurs in the vicinity of the turning latitude when meridional propagation approaches zero. As these waves violate the assumption that $\alpha \ll 1$ and behave in erratic and aphysical fashion, we dismiss them as inaccurate perturbation approximations; however, as will be shown in the next section we did find numerous wave tracings initiated on realistic two-dimensional fields for which the $\alpha \ll 1$ assumption was violated but wave behavior remained reasonable and uniquely matched to LBVE model solutions.

6. Realistic Two-Dimensional Atmospheric Fields

For the next set of experiments we again performed ray tracing and found forced solutions of the LBVE model using NCEP-NCAR reanalysis July-September 1949-2010 300hPa relative vorticity climatology (Kalnay et al. 1996) as the background. Gaussian divergence forcing was applied centered at 90°W and 5°N that extended 20°east and west but only 5° to the north and south of this center. This forcing mimics the position of precipitation anomalies associated with boreal summer El Niño events (Shaman and Tziperman 2007).

For ray tracing with this background, the full two-dimensional dispersion relation (Equation 3.2) now applies. For each ray initiation point and initial zonal wavenumber, there are 3 initial meridional wavenumber solutions. Per Equation 2.10c, the zonal wavenumber, $k$, changes as the Rossby wave propagates such that a real zonal wavenumber becomes complex, if the meridional wavenumber, $l$, is complex. The Rossby wave amplitude is again modified by the imaginary components of both the zonal and meridional wavenumbers (Equation 2.17).
Figure 8 presents solutions for initial integer zonal wavenumbers \( k = 1 \) through \( k = 12 \) at all sites between 0-10N and 110-70W. A total of 2268 waves are shown. The trajectories of the real and complex wavenumber ray tracings clearly diverge. The real wavenumber stationary barotropic Rossby waves remain trapped within the equatorial waveguide and propagate westward, whereas the complex wavenumber stationary barotropic Rossby waves typically escape the equatorial region and propagate into the subtropics of both the northern and southern hemisphere.

Figure 9 shows separately the ray tracing solutions for initial zonal wavenumber \( k = 4 \), \( k = 5 \) and \( k = 6 \). Many of the complex wavenumber stationary barotropic Rossby wave ray tracings move along trajectories that fall into three general groupings. The first grouping propagates northward to the region of subtropical convergence north of the forcing area and then propagates westward. Some of these rays propagate farther north and then move northeastward over the continental U.S. The second grouping moves north or northeast over the Caribbean. Some of these rays propagate east-north-east over Cuba and a few, by day 15, extend into the North African Asian jet. The third grouping propagates southward to about 10°S and then moves east.

The LBVE simulation was performed again with the same background and forcing, but with a sponge layer at 150°E to the west of the forcing region, per Shaman and Tziperman (2007). This sponge layer damps out westward propagating Rossby waves and isolates the response due to eastward propagating waves. This simulation reveals anomalies east of Cuba, west of Spain, and over and east of Brazil that arise within the LBVE model due to eastward propagating disturbances (Figure 10). These vorticity responses are consistent with complex wavenumber ray tracing trajectories. More specifically, complex Rossby wave ray tracings
can be seen traversing these regions (Figures 8 and 9).

The different trajectory behaviors among the complex wavenumber Rossby wave ray tracings are also evident in plots of individual initiation sites (Figure 11). Individual trajectory lines in fact depict two waves, initialized complex conjugates, one growing, one decaying, which propagate identically through space. Complex wavenumber ray tracings initiated at 252.5°E, 8.69°N produce rays that propagate northward into the zone of subtropical convergence evident in the LBVE model solution (Figure 11a). Complex wavenumber ray tracings initiated at 275°E, 3.72°N propagate east-north-eastward over the LBVE model solution anomaly in the vicinity of Cuba (Figure 11b). These waves exhibit a greater tendency to behave erratically in trajectory as well as amplitude; however, a number of them are well-behaved. Complex wavenumber ray tracings initiated at 275°E, 1.24°N propagate either northward into the zone of subtropical convergence north of the forcing region or southward and then eastward over South America (Figure 11c).

Figures 12 and 13 present the latitude, amplitude and wavenumber evolution over the course of 15 days integration for two of the growing complex Rossby waves shown in Figure 11. The east-northeastward propagating wave with initial zonal wavenumber $k = 3$ (Figure 12) traverses Cuba and nears the isolated vorticity anomaly evident in the LBVE model solution (Figure 10) during hour 80 at which time the wave amplitude grows markedly. The traced wave peaks in amplitude around hour 100 while in the vicinity of the vorticity anomaly. During these hours of integration the wave behaves somewhat erratically as its amplitude grows precipitously: the wave jumps in latitude and wavenumbers shift or change sign. In addition, as for many of the ray tracings performed on this 2-D JAS background, the imaginary and real wavenumber components are the same order of magnitude such that
\( \alpha \approx 1 \), rather than \( \alpha \ll 1 \), which violates the approximation used to insure that rays remain close to the real plane. However, these complex wavenumber Rossby waves traverse the region east of Cuba and exhibit rapid growth in this area, both of which behaviors are consistent with the LBVE solution showing an isolated anomaly east of Cuba that results from an eastward propagating signal (Figure 10b). In contrast, no real wavenumber Rossby waves traverse this region (Figure 8).

While this ray tracing amplitude and trajectory solution does not clearly corroborate the isolated anomaly finding of the LBVE model, the peak of amplitude near the anomaly is intriguing, and perhaps indicative of an area of heightened wave-mean flow interaction or wave breaking. Furthermore, other waves traversing the area also grow in amplitude in this same region and qualitatively identify the anomaly region (Figure 11).

The northward propagating wave with initial zonal wavenumber \( k = 4 \) (Figure 13) grows continuously during the 15 day integration. The trajectory and amplitude of this ray tracing vary smoothly, yet for the meridional wavenumber \( l_i \) and \( l_r \) are the same order of magnitude over the course of the integration. This wave propagates slowly into the region of subtropical convergence (Figure 11a).

For all the ray tracing presented here, the calculated amplitude does not take into account dissipative processes, nor does it define an initial wave amplitude, only relative changes in amplitude. Dissipation will decrease the wave amplitude; furthermore, the initial wave amplitude may be small. In addition, ray tracing does not account for nonlinear processes, such as wave-mean flow interaction or wavebreaking, which would likely dominate at large wave amplitudes. In spite of these limitations, both the trajectory and amplitude growth of the traced complex wavenumber Rossby waves are qualitatively consistent with the develop-
opment of the isolated vorticity anomalies seen over Cuba and the subtropical convergence zone north of the forcing region.

For further comparison, we performed a similar experiment during boreal winter over Indonesia. The LBVE model was linearized about 300hPa January-March (JFM) climatology and forced with divergence centered at 120°E, 5°N. Rossby wave ray tracings were initiated throughout the forcing region with initial integer zonal wavenumbers $k = 3$ through $k = 6$. For this region and season, there is less distinction between the real and complex stationary barotropic Rossby wave ray tracing trajectories (Figure 14); however, real wavenumber ray tracings show a greater tendency to propagate poleward and eastward along trajectories that follow the wavetrain of alternating positive and negative anomalies produced by the LBVE model. The complex rays tend to remain within the tropics and also show a greater tendency to cross into the southern hemisphere.

The similarity of the complex and real wavenumber Rossby wave trajectories initiated over Indonesia during boreal winter (Figure 14) differs from the tracings initiated over the eastern equatorial Pacific during boreal summer, which show a clearer divergence of complex and real wavenumber Rossby wave trajectory behaviors (Figure 8). Differences among these two tropical environments include differences in the background winds in both the tropics and extratropics. These changes may account for the different forced LBVE solutions, as well as the greater dispersion of complex and real wavenumber Rossby waves initiated over the eastern equatorial Pacific during JAS. The divergence of complex versus real wavenumber ray behaviors and its dependence on the background field is a topic for future study.
7. Discussion

Here we have shown that complex wavenumber stationary barotropic Rossby wave ray tracing can be used to track the amplitude and trajectory of propagating growing and decaying forced disturbances. Ray tracing amplitude solutions for these waves match exact solutions in a simple, idealized setting. Trajectories of these complex wavenumber Rossby wave ray tracings match forced solutions of a LBVE model linearized about a state of solid body rotation, as well as real wavenumber Rossby wave ray tracing trajectories.

Complex wavenumber Rossby wave ray tracing in a realistic zonally asymmetric atmosphere produces trajectories that are distinct from real wavenumber Rossby wave ray tracings and coincident with vorticity anomalies generated in similarly forced solutions of the LBVE model. Thus, it appears that inclusion of complex wavenumber stationary barotropic Rossby waves with the real wavenumber ray tracings provides a more complete description of the LBVE model solution. The amplitudes of these distinct complex wavenumber wave trajectories appear, on a qualitative level, to amplify in regions where the LBVE model vorticity solutions maximize. While the linearized ray tracing framework is somewhat simplistic, areas of marked Rossby wave amplitude growth could be indicative of localized wave-mean flow interaction or other growth or dissipation processes, such as wave breaking. Further investigation is needed to determine whether tracing of these complex wavenumber Rossby waves provides a means for identifying areas of such activity.

Care must be taken when using this ray tracing approach. Many of the integrations performed violate the assumption that $\alpha \ll 1$. In some instances this violation is associated with unusual and non-physical behaviors; however, many of the wave trajectories propagate
without large wavenumber changes over a single time step or radical shifts in direction or location. This finding indicates that the technique presented here can provide useful insight even when formal scaling assumptions are violated. Indeed, ray tracing of real wavenumber waves often violates WKB assumptions yet still prove informative.

For some waves, the imaginary components of the zonal and meridional wavenumbers of these complex Rossby waves could have offsetting effects on wave amplitude. That is, if the signs of the imaginary components of the 2 wavenumbers, \( k_i \) and \( l_i \), are opposite, one will act to increase wave amplitude while the other acts to decrease wave amplitude. The combined effect is that the amplitude of the wave may not grow or decay rapidly and thus may produce a signal like a real wavenumber Rossby wave. Dickinson and Clare (1973) noted that slowly decaying complex wavenumber Rossby waves could persist for some time in the atmosphere and ultimately extract momentum from the mean flow at considerable distances from their initial generation.

Future work is needed to determine the full utility of this methodology and complex wavenumber Rossby wave ray tracing in general. In particular, the factors controlling the dispersion of complex and real wavenumber ray trajectories need to be further investigated in order to determine the conditions in which these two wavenumber types characterize distinct behaviors.

Acknowledgments.

We thank Ian Eisenman and Michael Brenner for helpful discussions. This work was supported by NSF grants ATM-0917609 (JS), OCE-0424516 (RMS) and ATM-0917468 (ET).
APPENDIX

Solution to the Analytic System

The 4 waves satisfy two boundary conditions at $y = 0$, discontinuity matching of the quasi-geostrophic potential vorticity equation and continuity of pressure across the bound (i.e. $\psi$ is continuous). At $y = Y$, we have only 3 waves to consider, the 2 northward propagating Region II complex waves and the Region III northward propagating real wave. These 3 waves must satisfy the same 2 boundary conditions at $y = Y$.

The quasi geostrophic potential vorticity equation with $v = 0$ is

$$\left(\partial_t + u\partial_x\right)\nabla^2 \psi + \beta \psi_x + \bar{q}_x \psi_y = 0 \quad (A1)$$

Discontinuities in the effective $\beta$ occur at $y = 0$ at $y = Y$, which require use of a jump condition. To get this we integrate (A1) from $-\epsilon$ to $+\epsilon$ on either side of these bounds and equate the resulting equation with zero (Wang and Fyfe, 2000). All terms on the lhs but two vanish as $\epsilon \to 0$, yielding:

$$\left.\left(\omega + uk\right)\psi_y\right|_{-\epsilon}^{+\epsilon} + \left.\bar{q}_x \psi_y\right|_{-\epsilon}^{+\epsilon} = 0 \quad (A2)$$

which is our discontinuity matching condition (Equations 4.2b and d).

We consider stationary plane waves solutions of the form $\psi = A \exp(i(kx + ly))$ for the system described in Equations 4.2. This yields the equations:

$$1 + A_{Is} = A_{Ilg} + A_{Ild} \quad (A3a)$$
\[ l_{In} + l_{Is} A_{Is} = l_{IIg} A_{IIg} + l_{IIId} A_{IIId} \]  
\[ A_{III} = A_{IIg} e^{i l_{IIg} Y} + A_{IIId} e^{i l_{IIId} Y} \]  
\[ l_{III} A_{III} = l_{IIg} A_{IIg} e^{i l_{IIg} Y} + l_{IIId} A_{IIId} e^{i l_{IIId} Y} \]

For the solutions for a prescribed background \((u, v, \beta, \beta_T, \partial \bar{q} / \partial x)\) and initial zonal wavenumber and wave frequency, the meridional wavenumber of each wave \(l_{In}, l_{Is}, l_{IIg}, l_{IIId}, l_{III}\) are found using (3.2). Solving for the amplitudes, yields the following solutions

\[ A_{IIId} = \frac{l_{In} - l_{Is}}{(l_{Is} - l_{IIg})(l_{III} - l_{IIId}) e^{i(l_{IIId} - l_{IIg})Y} + (l_{III} - l_{IIg})} \]  
\[ A_{IIg} = \frac{A_{IIId}(l_{IIId} - l_{III}) - (l_{In} + l_{Is})}{l_{Is} - l_{IIg}} \]  
\[ A_{III} = A_{IIg} e^{i(l_{IIg} - l_{III})Y} + A_{IIId} e^{i(l_{IIId} - l_{III})Y} \]  
\[ A_{Is} = A_{IIg} + A_{IIId} - 1 \]

Ray tracing solutions are solved for using the methods described in Section 2. This tracing provides solutions for the change in wave amplitude as well as zonal translocation while traversing the Region II domain in the meridional direction. An analytic solution of Rossby wave zonal translocation within Region II is determined based on the ratio of the zonal and meridional wavenumbers and Region II extent, \(Y\) for comparison.
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Two example ray tracing solutions initiated at the same location ($18.62^\circ$N, $172.5^\circ$E) on solid body rotation with $u = 15 \text{m/s}$ at the equator. The left plot is for a ray tracing with initial zonal wavenumber $k = 5$; the right plot is for a ray tracing with initial zonal wavenumber $k = 5 + 0.01i$. Shown are a) wave latitude; b) wave amplitude; c) imaginary wavenumber parts; and d) real wavenumber parts during 15 days (360 hours) of integration.

As for Figure 4, pairs of ray tracings initiated within the forcing region with initial zonal wavenumbers $k = 5$ (blue lines) and $k = 5 + 0.01i$ (red lines). For these tracings the complex wavenumber Rossby waves remain at a poleward latitude rather than turn equatorward with their real counterparts. All waves are integrated for 15 days. Square black markers show the location along the real ray trajectory every 48 hours.
Two example ray tracing solutions initiated at the same location (3.72°N, 172.5°E) on solid body rotation with $u = 15m/s$ at the equator. The left plot is for a ray tracing with initial zonal wavenumber $k = 5$; the right plot is for a ray tracing with initial zonal wavenumber $k = 5 + 0.01i$. Shown are a) wave latitude; b) wave amplitude; c) imaginary wavenumber parts; and d) real wavenumber parts during 15 days (360 hours) of integration. The complex wavenumber wave behaves erratically in the vicinity of the turning latitude as the real meridional wavenumber nears zero.

Steady LBVE solution to Gaussian amplitude divergent forcing centered at 5°N, 270°E. Divergence maximizes in center at $-3 \times 10^{-6}s^{-1}$. The background is JAS 1949-2010 300mb climatology that has been zonally smoothed to wavenumber 8. Rossby wave ray tracings initiated throughout the forcing region with initial integer zonal wavenumbers $k = 1$ through $k = 12$ are shown. a) All real wavenumber ray tracings. b) All complex wavenumber ray tracings.

As in Figure 8, but for ray tracings initiated with initial zonal wavenumber: a) $k = 4$ only; b) $k = 5$ only; c) $k = 6$ only. Black lines are real Rossby wave ray tracings; green lines are complex Rossby wave ray tracings.
LBVE solution to Gaussian amplitude divergent forcing centered at 5°N, 270°E. Divergence maximizes in center at \(-3 \times 10^{-6} \text{s}^{-1}\). The background is JAS 1949-2010 300mb climatology that has been zonally smoothed to wavenumber 8. a) full solution, as in Figure 8. b) Same as top, but a sponge layer has been applied at 150°E to preclude Rossby wave propagation westward from the Pacific basin. Contour are \(\pm 0.2 \times 10^{-6} \text{s}^{-1}\), \(\pm 0.4 \times 10^{-6} \text{s}^{-1}\), \(\pm 0.6 \times 10^{-6} \text{s}^{-1}\), \(\pm 1 \times 10^{-6} \text{s}^{-1}\), \(\pm 2 \times 10^{-6} \text{s}^{-1}\), and \(-4 \times 10^{-6} \text{s}^{-1}\). Negative contours are dashed.

Steady LBVE solution to Gaussian amplitude divergent forcing centered at 5°N, 270°E. Divergence maximizes in center at \(-3 \times 10^{-6} \text{s}^{-1}\). The background is JAS 1949-2010 300mb climatology that has been zonally smoothed to wavenumber 8. A sponge layer has been applied centered at 150°E that precludes Rossby wave propagation westward from the Pacific basin. Rossby wave ray tracings initiated at 3 individual sites with initial integer zonal wavenumbers \(k = 1\) through \(k = 12\) are shown. The sites are a) 252.5°E, 8.69°N b) 275°E, 3.72°N c) 275°E, 1.24°N. Black lines are real Rossby wave ray tracings; red lines are complex Rossby wave ray tracings.

Plot of the wave latitude, amplitude (relative to initial value), imaginary wavenumbers, and real wavenumbers of the growing ray initiated with zonal wavenumber \(k = 3\) at 275°E, 1.24°N. a) wave latitude; b) wave amplitude; c) imaginary wave numbers; d) real wavenumbers.

As for Figure 12, but for the growing ray initiated with zonal wavenumber \(k = 4\) at 252.5°E, 8.69°N.
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Region III
\[ \frac{\partial q}{\partial y} = \beta \]
\[ k, l_{m+} \]

Region II
\[ \frac{\partial q}{\partial y} = \beta + \beta_r \]
\[ k, l_{ld} \quad k, l_{rd} \]

Region I
\[ \frac{\partial q}{\partial y} = \beta \]
\[ k, l_{t+} \quad k, l_{t-} \]

Fig. 1. Schematic of the infinite \( \beta \)-plane used for comparing ray traced and exact amplitude solutions.
Fig. 2. Wave height solutions for Rossby waves on infinite β-plane, as described in text. Solutions are shown for the 3 Rossby waves at \(y = Y\) as a function of Region II domain size.
Fig. 3. Comparison of analytic and ray traced amplitude solutions of Region II decaying mode at $y = Y$ for a range of Region II domain sizes. Solutions are for the same system as in Figure 2: $u = 20\text{m/s}$, $v = 0\text{m/s}$, $\beta = 2 \times 10^{-11}\text{s}^{-1}\text{m}^{-1}$, $\beta + \beta_T = -3.26 \times 10^{-11}\text{s}^{-1}\text{m}^{-1}$, $\partial \bar{q} / \partial x = -4.5 \times 10^{-11}\text{s}^{-1}\text{m}^{-1}$, and zonal wavenumber $k = 5$. The thick gray line shows the analytic solution; the thin black line is the ray traced solution.
FIG. 4. LBVE steady solution to Gaussian amplitude divergent forcing centered at 0°N, 180°E. Divergence maximizes in center at $-3 \times 10^{-6}\text{s}^{-1}$. The background is in solid body rotation that maximizes at 15m/s at the equator. Planetary rotation is as for earth. Superimposed on this solution are ray tracings initiated within the forcing region with initial zonal wavenumbers $k = 5$ (blue lines) and $k = 5 + 0.01i$ (red lines) initiated within the forcing region. All waves are integrated for 15 days. Square black markers show the location along the real ray trajectory every 48 hours. Black contours show the LBVE solution with contour intervals of $0.5 \times 10^{-6}\text{s}^{-1}$ to $\pm 2 \times 10^{-6}\text{s}^{-1}$ and $2 \times 10^{-6}\text{s}^{-1}$ for larger magnitudes. Negative contours are dashed; the zero contour is omitted.
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Fig. 8. Steady LBVE solution to Gaussian amplitude divergent forcing centered at 5°N, 270°E. Divergence maximizes in center at $-3 \times 10^{-6}\, \text{s}^{-1}$. The background is JAS 1949-2010 300mb climatology that has been zonally smoothed to wavenumber 8. Rossby wave ray tracings initiated throughout the forcing region with initial integer zonal wavenumbers $k = 1$ through $k = 12$ are shown. a) All real wavenumber ray tracings. b) All complex wavenumber ray tracings.
Fig. 9. As in Figure 8, but for ray tracings initiated with initial zonal wavenumber: a) $k = 4$ only; b) $k = 5$ only; c) $k = 6$ only. Black lines are real Rossby wave ray tracings; green lines are complex Rossby wave ray tracings.
Fig. 10. LBVE solution to Gaussian amplitude divergent forcing centered at 5°N, 270°E. Divergence maximizes in center at $-3 \times 10^{-6} s^{-1}$. The background is JAS 1949-2010 300mb climatology that has been zonally smoothed to wavenumber 8. a) full solution, as in Figure 8. b) Same as top, but a sponge layer has been applied at 150°E to preclude Rossby wave propagation westward from the Pacific basin. Contour are $\pm 0.2 \times 10^{-6} s^{-1}$, $\pm 0.4 \times 10^{-6} s^{-1}$, $\pm 0.6 \times 10^{-6} s^{-1}$, $\pm 1 \times 10^{-6} s^{-1}$, $\pm 2 \times 10^{-6} s^{-1}$, and $-4 \times 10^{-6} s^{-1}$. Negative contours are dashed.
Fig. 11. Steady LBVE solution to Gaussian amplitude divergent forcing centered at 5°N, 270°E. Divergence maximizes in center at $-3 \times 10^{-6} \text{s}^{-1}$. The background is JAS 1949-2010 300mb climatology that has been zonally smoothed to wavenumber 8. A sponge layer has been applied centered at 150°E that precludes Rossby wave propagation westward from the Pacific basin. Rossby wave ray tracings initiated at 3 individual sites with initial integer zonal wavenumbers $k = 1$ through $k = 12$ are shown. The sites are a) 252.5°E, 8.69°N b) 275°E, 3.72°N c) 275°E, 1.24°N. Black lines are real Rossby wave ray tracings; green lines are complex Rossby wave ray tracings.
Fig. 12. Plot of the wave latitude, amplitude (relative to initial value), imaginary wavenumbers, and real wavenumbers of the growing ray initiated with zonal wavenumber $k = 3$ at 275°E, 1.24°N. a) wave latitude; b) wave amplitude; c) imaginary wave numbers; d) real wavenumbers.
Fig. 13. As for Figure 12, but for the growing ray initiated with zonal wavenumber $k = 4$ at 252.5°E, 8.69°N.
Fig. 14. Steady LBVE solution to Gaussian amplitude divergent forcing centered at 5°N, 120°E. Divergence maximizes in center at $-3 \times 10^{-6}$ s$^{-1}$. The background is JFM 1949-2010 300mb climatology that has been zonally smoothed to wavenumber 8. Rossby wave ray tracings initiated throughout the forcing region with initial integer zonal wavenumbers $k = 3$ through $k = 6$ are shown. Top) Real wavenumber ray tracings. Bottom) Complex wavenumber ray tracings.